QUOTIENT TRIANGULATED CATEGORIES

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Dedicated to Professor Zhe-Xian Wan on the occasion of his eightieth birthday

Abstract. For a self-orthogonal module T, the relation between the quotient triangulated category $D^b(A)/K^b(\mathrm{add}T)$ and the stable category of the Frobenius category of T-Cohen-Macaulay modules is investigated. In particular, for a Gorenstein algebra, we get a relative version of the description of the singularity category due to Happel. Also, the derived category of a Gorenstein algebra is explicitly given, inside the stable category of the graded module category of the corresponding trivial extension algebra, via Happel's functor $F : D^b(A) \longrightarrow T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$.

Introduction

Throughout, A is a finite-dimensional associative algebra over a field k, A-mod the category of finite-dimensional left A-modules, and $D^b(A)$ the bounded derived category of A-mod. Let $K^b(A\text{-inj})$ and $K^b(A\text{-proj})$ be the bounded homotopy categories of injective, and projective A-modules, respectively. Viewing them as thick (épaisse; see [V1]) triangulated subcategories of $D^b(A)$, one has the quotient triangulated categories:

$$\mathcal{D}_I(A) := D^b(A)/K^b(A\text{-inj}) \quad \text{and} \quad \mathcal{D}_P(A) := D^b(A)/K^b(A\text{-proj}).$$

Note that $\mathcal{D}_I(A) = 0$ (resp. $\mathcal{D}_P(A) = 0$) if and only if gl.dimA $< \infty$. In the same way, for an algebraic variety X one has the quotient triangulated category $\mathbf{D}_{Sg}(X) := \mathbf{D}^b(coh(X))/\mathsf{perf}(X)$, where $\mathbf{D}^b(coh(X))$ is the bounded derived category of coherent sheaves on X, and $\mathsf{perf}(X)$ is its full subcategory of perfect complexes. Note that $\mathbf{D}_{Sg}(X) = 0$ if and only if X is smooth. Thus, for an algebra A of infinite global dimension, or a singular variety X, it is of interest to investigate $\mathcal{D}_I(A)$, $\mathcal{D}_P(A)$, and $\mathbf{D}_{Sg}(X)$. These quotient triangulated categories become an important topic in algebraic geometry and representation theory of algebras through the work of Buchweitz [Buc], Keller-Vossieck [KV1], Rickard [Ric2], Happel [Hap2], Beligiannis [Bel], Jørgensen [J], Orlov [O1], [O2], Krause [Kr], Krause-Iyengar [KI], and others, as they measure the complexity of possible singularities. In particular, they are called the singularity categories in [O1]; in [Ric2] (Theorem 2.1) it was proved that for a self-injective algebra A, $\mathcal{D}_P(A)$ is triangle-equivalent to the stable module category of A modulo the projectives; and in [Hap2] (Theorem 4.6) it was proved that for a Gorenstein algebra A, $\mathcal{D}_P(A)$ is triangle-equivalent to the stable category of the Frobenius category of Cohen-Macaulay modules.

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For a self-orthogonal A-module T (i.e., $\mathrm{Ext}^i_A(T,T)=0$ for each $i\geq 1$), let add$T$ denote the full subcategory of A-mod whose objects are the direct summands of finite direct sum of copies of T. Then $K^b(\mathrm{add}T)$ is a triangulated subcategory of $D^b(A)$ ([Hap1], p.103). Since both $K^b(\mathrm{add}T)$ and $D^b(A)$ are Krull-Schmidt categories (see [KV2], or [BD]), i.e., each object can be uniquely decomposed into a direct sum of (finitely many) indecomposables and indecomposables have local endomorphism rings, it follows that $K^b(\mathrm{add}T)$ is closed under direct summands, that is, $K^b(\mathrm{add}T)$ is thick in $D^b(A)$ (see Proposition 1.3 in [Ric2]; or [V2]), and hence one has the quotient triangulated category $D^b(A)/K^b(\mathrm{add}T)$. In the view of the tilting theory (see e.g. [HR], [Rin1], [Hap1], [AR1], [M]), $\mathcal{D}_I(A)$ (resp., $\mathcal{D}_P(A)$) is just the special case of $\mathcal{D}_T(A)$ when T is a generalized cotilting module (resp., T is a generalized tilting module). This encourages us to look at $\mathcal{D}_I(A)$ and $\mathcal{D}_P(A)$ in terms of generalized cotilting and tilting modules, respectively, and to understand $\mathcal{D}_T(A)$ for self-orthogonal modules T, in general. If $(A, {}_AT_B, B)$ is a generalized (co)tilting triple, then by a theorem due to Happel ([Hap1], Theorem 2.10, p.109, for the finite global dimension case), and due to Cline, Parshall, and Scott ([CPS], Theorem 2.1, for general case. See also Rickard [Ric1], Theorem 6.4, in terms of tilting complexes), this investigation permits us to understand the singularity category $\mathcal{D}_P(B)$ in terms of ${}_AT$.

For a self-orthogonal module T, we study in Section 2 the relation between the quotient triangulated category $D^b(A)/K^b(\mathrm{add}T)$ and the stable category of the Frobenius category $\alpha(T)$ of T-Cohen-Macauley modules (see 2.1 for the definition of this terminology). In particular, for a Gorenstein algebra, we get the relative version of the description of the singularity category due to Happel ([Hap2], Theorem 4.6). See Theorem 2.5.

Denote by $T(A) := A\oplus D(A)$ the trivial extension of A, where $D=\mathrm{Hom}_k(-,k)$. It is $\mathbb{Z}$-graded with deg$A=0$ and deg$D(A)=1$. Denote by $T(A)^{\mathbb{Z}}$-mod the category of finite-dimensional $\mathbb{Z}$-graded $T(A)$-modules with morphisms of degree 0. This is a Frobenius abelian category, and hence its stable category $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$ modulo projectives is a triangulated category. Happel has constructed a fully faithful exact functor $F: D^b(A)\longrightarrow T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$ ([Hap1], p.88, plus p.64); and F is dense if and only if gl.dimA $<\infty$ ([Hap3]). Consider the natural embedding i : A-mod $\hookrightarrow T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$ such that each A-module M is a graded $T(A)$-module concentrated at degree 0, which is the restriction of F. Denote by $\mathcal{N}$, $\mathcal{M}_P$, and $\mathcal{M}_I$ the triangulated subcategories of $T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$ generated by A-mod, A-proj, and A-inj, respectively. Since both $T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$ and $\mathcal{N}$ are Krull-Schmidt, it follows that $\mathcal{N}$ is closed under direct summands, that is, $\mathcal{N}$ is thick in $T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$; and so are $\mathcal{M}_P$ and $\mathcal{M}_I$. So, one has the quotient triangulated categories:

$$T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}/\mathcal{N}, \quad T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}/\mathcal{M}_I, \quad \text{and} \quad T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}/\mathcal{M}_P.$$

Then Happel's theorem above reads as: there are equivalences of triangulated categories

$$F: D^b(A)\simeq\mathcal{N}, \quad F: K^b(A\text{-inj})\simeq\mathcal{M}_I, \quad \text{and} \quad F: K^b(A\text{-proj})\simeq\mathcal{M}_P;$$

and $T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}/\mathcal{N}=0$ if and only if gl.dim $A<\infty$.

It is then of interest to study these quotient triangulated categories. In Section 3 we only take the first step by giving an explicit description of the bounded derived category of a Gorenstein algebra A inside $T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$, via Happel's functor above. See Theorem 3.1.

1. **Preliminaries**

1.1. An algebra A is Gorenstein if proj.dim ${}_AD(A_A)<\infty$ and inj.dim ${}_AA<\infty$. Self-injective algebras and algebras of finite global dimension are Gorenstein; the tensor product $A\otimes_k B$ is Gorenstein if and only if so are A and B ([AR2], Proposition 2.2). Note that

A is Gorenstein if and only if $K^b(A\text{-proj}) = K^b(A\text{-inj})$ inside $D^b(A)$ ([Hap2], Lemma 1.5). Thus, by Theorem 6.4 and Proposition 9.1 in [Ric1], if the algebras A and B are derived equivalent, then A is Gorenstein if and only if so is B. Also, cluster-tilted algebras are Gorenstein ([KR]).

1.2. For basics on triangulated categories and derived categories we refer to [Har] and [V1]. Following [BBD], the shift functor in a triangulated category is denoted by [1]. Recall that, by definition, triangulated subcategories are full subcategories. By a multiplicative system, we will always mean a multiplicative system compatible with the triangulation. For a multiplicative system S of a triangulated category $\mathcal{K}$, we refer to [Har] (see also [V1] and [I]) for the construction of the quotient triangulated category $S^{-1}\mathcal{K}$ via localization, in which morphisms are given by right fractions (if one uses left fractions then one gets a quotient triangulated category isomorphic to $S^{-1}\mathcal{K}$).

Let $\mathcal{A}$ be an abelian category, $\mathcal{B}$ a full subcategory of $\mathcal{A}$, and $\varphi : K^b(\mathcal{B}) \longrightarrow D^b(\mathcal{A})$ the composition of the embedding $K^b(\mathcal{B}) \hookrightarrow K^b(\mathcal{A})$ and the localization functor $K^b(\mathcal{A}) \longrightarrow D^b(\mathcal{A})$. If φ is fully faithful, then $K^b(\mathcal{B})$ is a triangulated subcategory of $D^b(\mathcal{A})$.

Applying this to add$T$ with T a self-orthogonal A-module, we know by Lemma 2.1 in [Hap1], p.103, that $K^b(\mathrm{add}T)$ is a triangulated subcategory of $D^b(A)$. If T is a generalized tilting module, then $K^b(\mathrm{add}T) = K^b(A\text{-proj})$ in $D^b(A)$ (in fact, for any projective module P and $T' \in \mathrm{add}T$, we have $P \in K^b(\mathrm{add}T)$ and $T' \in K^b(A\text{-proj})$, in $D^b(A)$. Then the assertion follows from the fact that $K^b(A\text{-proj})$ and $K^b(\mathrm{add}T)$ are the triangulated subcategories of $D^b(A)$ generated by add$A$ and by add$T$, respectively).

1.3. An exact category $\mathcal{A}$ is a full subcategory of an abelian category, closed under extensions and direct summands, together with the exact structure given by the set of all the short exact sequences of the ambient abelian category with terms in $\mathcal{A}$. Such exact sequences will be referred as admissible exact sequences (see Quillen [Q]; or conflations in the sense of Gabriel-Roiter, see e.g. Appendix A in [K1]). A Frobenius category is an exact category in which there are enough (relative) injective objects and (relative) projective objects, such that the injective objects coincide with the projective objects. For the reason requiring that $\mathcal{A}$ is closed under direct summands see Lemma 1.1 below. Compare p.10 in [Hap1], Appendix A in [K1], and [Q]. Denote by $\underline{\mathcal{A}}$ its stable category. For a morphism $u: X \longrightarrow Y$ in $\mathcal{A}$, denote its image in $\underline{\mathcal{A}}$ by $\underline{u}$.

Lemma 1.1. *Let $\mathcal{A}$ be a Frobenius category. Then $X \simeq Y$ in $\underline{\mathcal{A}}$ if and only if there are injective objects I and J such that $X \oplus J \simeq Y \oplus I$ in $\mathcal{A}$.*

Proof. This is well-known. We include a proof for convenience. Let $\underline{f} : X \longrightarrow Y$ be an isomorphism in $\underline{\mathcal{A}}$. Then there exist an injective object I, $g : Y \longrightarrow X$, $a : X \longrightarrow I$ and $b : I \longrightarrow X$, such that $(g, -b) \circ \binom{f}{a} = \mathrm{Id}_X$. Thus there exists J in $\mathcal{A}$ (here we need $\mathcal{A}$ being closed under direct summands) and $h : X \oplus J \simeq Y \oplus I$, such that $\binom{f}{a} = h\binom{1}{0}$ in $\mathcal{A}$. Thus $\underline{\binom{1}{0}} : X \longrightarrow X \oplus J$ is an isomorphism in $\underline{\mathcal{A}}$. Then by a matrix calculation we have $\underline{\mathrm{Id}_J = 0}$, i.e., J is an injective object. ■

Let $\mathcal{A}$ be a Frobenius category. Recall the triangulated structure in $\underline{\mathcal{A}}$ from [Hap1], Chapter 1, Section 2. The shift functor $[1] : \underline{\mathcal{A}} \longrightarrow \underline{\mathcal{A}}$ is defined such that for $X \in \mathcal{A}$,

$$0 \longrightarrow X \xrightarrow{i_X} I(X) \xrightarrow{\pi_X} X[1] \longrightarrow 0 \tag{1.1}$$

is an admissible exact sequence in $\mathcal{A}$ with $I(X)$ an injective object. By Lemma 1.1, if $X \simeq Y$ in $\underline{\mathcal{A}}$ then $X[1] \simeq Y[1]$ in $\underline{\mathcal{A}}$, and the object $X[1]$ in $\underline{\mathcal{A}}$ does not depend on the choice of (1.1). For $u: X \longrightarrow Y$ in $\mathcal{A}$, the standard triangle $X \xrightarrow{\underline{u}} Y \xrightarrow{\underline{v}} C_u \xrightarrow{\underline{w}} X[1]$ in $\underline{\mathcal{A}}$ is defined by the pushout diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "i_X"] \arrow[d, "u"] & I(X) \arrow[r, "\pi_X"] \arrow[d, "\bar{u}"] & X[1] \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Y \arrow[r, "v"] & C_u \arrow[r, "w"] & X[1] \arrow[r] & 0;
\end{tikzcd}$$

and then the distinguished triangles in $\underline{\mathcal{A}}$ are defined to be the triangles isomorphic to the standard ones. We need the following fact in [Hap1], p.22.

Lemma 1.2. *Distinguished triangles in $\underline{\mathcal{A}}$ are just given by short exact sequences in $\mathcal{A}$.*

More precisely, let $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ be an admissible exact sequence in $\mathcal{A}$. Then $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} X[1]$ is a distinguished triangle in $\underline{\mathcal{A}}$, where w is an $\mathcal{A}$-map such that the following diagram is commutative (note that any two such maps w and w' give the isomorphic triangles)

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, dotted, "\sigma"] & Z \arrow[r] \arrow[d, dotted, "w"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_X"] & I(X) \arrow[r, "\pi_X"] & X[1] \arrow[r] & 0.
\end{tikzcd} \tag{1.2}$$

Conversely, let $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{-w'} X'[1]$ be a distinguished triangle in $\underline{\mathcal{A}}$. Then there is an admissible exact sequence $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ in $\mathcal{A}$, such that the induced distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} X[1]$ is isomorphic to the given one, where w is an $\mathcal{A}$-map such that (1.2) *is commutative.*

Proof. We include a proof of the second part for convenience. Let $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{-w'} X'[1]$ be a distinguished triangle in $\underline{\mathcal{A}}$. Then it is isomorphic to a standard triangle $X \xrightarrow{u} Y \xrightarrow{v} C_u \xrightarrow{w} X[1]$ in $\underline{\mathcal{A}}$, with an admissible exact sequence in $\mathcal{A}$:

$$0 \longrightarrow X \xrightarrow{\binom{u}{i_X}} Y \oplus I(X) \xrightarrow{(v,\ -\bar{u})} C_u \longrightarrow 0 \tag{1.3}$$

and the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\binom{u}{i_X}"] \arrow[d, equal] & Y \oplus I(X) \arrow[r, "(v,\ -\bar{u})"] \arrow[d, "(0,1)"] & C_u \arrow[r] \arrow[d, "-w"] & 0 \\
0 \arrow[r] & X \arrow[r, "i_X"] & I(X) \arrow[r, "\pi_X"] & X[1] \arrow[r] & 0.
\end{tikzcd}$$

This implies that the distinguished triangle induced by (1.3) is isomorphic to the standard triangle. ■

Lemma 1.3. *Let $\mathcal{A}$ be a Frobenius category. Then there is a bijection between the class of the full subcategories $\mathcal{B}$ of $\mathcal{A}$, where $\mathcal{B}$ contains all the injective objects of $\mathcal{A}$, such that if two terms in an admissile exact sequence in $\mathcal{A}$ lie in $\mathcal{B}$, then the third term also lies in $\mathcal{B}$, and the class of triangulated subcategories of $\underline{\mathcal{A}}$.*

Proof. If $\mathcal{B}$ is such a full subcategory of $\mathcal{A}$, then by Lemmas 1.2 and 1.1 $\underline{\mathcal{B}}$ is a triangulated subcategory of $\underline{\mathcal{A}}$. Conversely, let $\mathcal{D}$ be a triangulated subcategory of $\underline{\mathcal{A}}$. Set

$$\mathcal{B} := \{X \in \mathcal{A} \mid \text{there exists } Y \in \mathcal{D} \text{ such that } X \simeq Y \text{ in } \underline{\mathcal{A}}\}.$$

Then $\mathcal{D} = \underline{\mathcal{B}}$. Since $\mathcal{D}$ contains zero object, it follows that $\mathcal{B}$ contains all the injective objects of $\mathcal{A}$; and by Lemma 1.1 $\mathcal{B}$ has the required property. If $\mathcal{B}$ and $\mathcal{B}'$ are two such a different full subcategories of $\mathcal{A}$, then by Lemma 1.1 $\underline{\mathcal{B}}$ and $\underline{\mathcal{B}'}$ are also different. ■

2. Quotient triangulated category $D^b(A)/K^b(\mathrm{add}T)$

Throughout the section, T is a self-orthogonal A-module. All subcategories will be assumed to be closed under isomorphisms and finite direct sums. However, following [Rin2], we do not assume that they are closed under direct summands.

2.1. Consider the following full subcategories of A-mod introduced in [AR1]:

$$T^{\perp} := \{X \mid \mathrm{Ext}^i_A(T, X) = 0, \ \forall\ i \geq 1\},$$

$$\mathrm{add}^{\sim}T := \{X \mid \exists \text{ an exact sequence } \cdots \longrightarrow T^{-i} \longrightarrow \cdots T^0 \longrightarrow X \longrightarrow 0,\ T^{-i} \in \mathrm{add}T, \forall\ i\},$$

$${}_T\mathcal{X} := \{X \mid \exists \text{ an exact sequence } \cdots \longrightarrow T^{-i} \xrightarrow{d^{-i}} T^{-(i-1)} \longrightarrow \cdots \xrightarrow{d^{-1}} T^0 \xrightarrow{d^0} X \longrightarrow 0,$$
$$T^{-i} \in \mathrm{add}T, \ \ \mathrm{Ker}d^{-i} \in T^{\perp}, \ \forall\ i \geq 0\},$$

and

$$\mathrm{add}^{\wedge}T := \{X \mid \exists \text{ an exact sequence } 0 \longrightarrow T^{-n} \longrightarrow \cdots T^0 \longrightarrow X \longrightarrow 0,\ T^{-i} \in \mathrm{add}T, \ \ \forall\ i\ \}.$$

By dimension-shifting we have $\mathrm{add}^{\wedge}T \subseteq {}_T\mathcal{X} \subseteq \mathrm{add}^{\sim}T \cap T^{\perp}$. Note that ${}_T\mathcal{X} = \mathrm{add}^{\sim}T$ if and only if $\mathrm{add}^{\sim}T \subseteq T^{\perp}$ (For the "if part", note that $\mathrm{Ker}d^{-i}$ is still in $\mathrm{add}^{\sim}T$, and hence in $T^{\perp}$). If T is exceptional (i.e., proj.dim $T < \infty$ and T is self-orthogonal), then $\mathrm{add}^{\sim}T \subseteq T^{\perp}$, and hence ${}_T\mathcal{X} = \mathrm{add}^{\sim}T$.

If T is a generalized tilting module (i.e., T is exceptional, and there is an exact sequence $0 \longrightarrow {}_AA \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^n \longrightarrow 0$ with each $T^i \in \mathrm{add}T$), then $\mathrm{add}^{\sim}T = T^{\perp}$, and hence ${}_T\mathcal{X} = \mathrm{add}^{\sim}T = T^{\perp}$. (In fact, by the theory of generalized tilting modules, $X \in T^{\perp}$ can be generated by T, see [M], Lemma 1.8; and then by using a classical argument in [HR], p.408, one can prove $X \in \mathrm{add}^{\sim}T$ by induction.)

If gl.dim$A < \infty$ then $\mathrm{add}^{\wedge}T = {}_T\mathcal{X} = \mathrm{add}^{\sim}T$ for any self-orthogonal module T. For this it suffices to prove ${}_T\mathcal{X} \subseteq \mathrm{add}^{\wedge}T$. This follows from

$$\mathrm{Ext}^1_A(\mathrm{Ker}d^{-(i-1)}, \mathrm{Ker}d^{-i}) \cong \mathrm{Ext}^2_A(\mathrm{Ker}d^{-(i-2)}, \mathrm{Ker}d^{-i}) \cong \cdots \cong \mathrm{Ext}^i_A(\mathrm{Ker}d^0, \mathrm{Ker}d^{-i}) = 0,$$

where $i \gg 0$.

Dually, we have the concept of a generalized cotilting module, full subcategories of A-mod: ${}^{\perp}T$, $\mathrm{add}_{\sim}T$, $\mathcal{X}_T$, $\mathrm{add}_{\wedge}T$, and the corresponding facts.

We define the category of T-Cohen-Macaulay modules to be the subcategory $\mathfrak{a}(T) := \mathcal{X}_T \cap {}_T\mathcal{X}$. One may compare it with the definition of the category Cohen-Macaulay modules given in [Hap2] and [AR2]. By Proposition 5.1 in [AR1], both subcategories $\mathcal{X}_T$ and ${}_T\mathcal{X}$ are closed under extensions and direct summands, and thus so is $\alpha(T)$. One can easily verify that $\mathfrak{a}(T)$ is a Frobenius category, where add$T$ is exactly the full subcategory of all the (relatively) projective and injective objects.

2.2. For $M, N \in A$-mod, let $T(M, N)$ denote the subspace of A-maps from M to N which factor through add$T$. The following lemma seems to be of independent interest. We give an explicit proof by using calculus of fractions.

Lemma 2.1. *Let T be a self-orthogonal module. If $M \in \mathcal{X}_T$ and $N \in T^{\perp}$, then there is a canonical isomorphism of k-spaces*

$$\mathrm{Hom}_A(M, N)/T(M, N) \simeq \mathrm{Hom}_{D^b(A)/K^b(\mathrm{add}T)}(M, N).$$

Proof. In what follows, a doubled arrow means a morphism belonging to the saturated multiplicative system, determined by the thick triangulated subcategory $K^b(\mathrm{add}T)$ of $D^b(A)$ (see [V1], [Har], or [I]). A morphism from M to N in $D^b(A)/K^b(\mathrm{add}T)$ is denoted by right fraction $a/s: \ M \overset{s}{\Longleftarrow} Z^\bullet \xrightarrow{a} N$, where $Z^\bullet \in D^b(A)$. Note that the mapping cone $\mathrm{Con}(s)$ lies in $K^b(\mathrm{add}T)$. We have a distinguished triangle in $D^b(A)$

$$Z^\bullet \overset{s}{\Longrightarrow} M \longrightarrow \mathrm{Con}(s) \longrightarrow Z^\bullet[1]. \tag{2.1}$$

Consider the k-map $G: \mathrm{Hom}_A(M,N) \longrightarrow \mathrm{Hom}_{D^b(A)/K^b(\mathrm{add}T)}(M,N)$, given by $G(f) = f/\mathrm{Id}_M$. First, we prove that G is surjective. By $M \in \mathcal{X}_T$ we have an exact sequence

$$0 \longrightarrow M \xrightarrow{\varepsilon} T^0 \xrightarrow{d^0} T^1 \xrightarrow{d^1} \cdots \longrightarrow T^n \xrightarrow{d^n} \cdots$$

with $\mathrm{Im}d^i \in {}^\perp T$, $\forall\, i \geq 0$. Then M is isomorphic in $D^b(A)$ to the complex $T^\bullet := 0 \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots$, and then isomorphic to the complex $0 \longrightarrow T^0 \longrightarrow \cdots \longrightarrow T^{l-1} \longrightarrow \mathrm{Ker}d^l \longrightarrow 0$ for each $l \geq 1$. The last complex induces a distinguished triangle in $D^b(A)$

$$\sigma^{<l}T^\bullet[-1] \longrightarrow \mathrm{Ker}d^l[-l] \overset{s'}{\Longrightarrow} M \xrightarrow{\varepsilon} \sigma^{<l}T^\bullet, \tag{2.2}$$

where $\sigma^{<l}T^\bullet = 0 \longrightarrow T^0 \longrightarrow T^1 \longrightarrow \cdots \longrightarrow T^{l-1} \longrightarrow 0$, and the mapping cone of s' lies in $K^b(\mathrm{add}T)$. Since $\mathrm{Ker}d^l \in {}^\perp T$ and $\mathrm{Con}(s) \in K^b(\mathrm{add}T)$, it follows that there exists $l_0 \gg 0$ such that for each $l \geq l_0$

$$\mathrm{Hom}_{D^b(A)}(\mathrm{Ker}d^l[-l], \mathrm{Con}(s)) = 0.$$

(To see this, let $\mathrm{Con}(s)$ be of the form $0 \longrightarrow W^{-t'} \longrightarrow \cdots \longrightarrow W^t \longrightarrow 0$ with $t', t \geq 0$, and each $W^i \in \mathrm{add}T$. Consider the distinguished triangle in $D^b(A)$

$$\sigma^{<t}\mathrm{Con}(s)[-1] \longrightarrow W^t[-t] \longrightarrow \mathrm{Con}(s) \longrightarrow \sigma^{<t}\mathrm{Con}(s),$$

Take l_0 to be $t+1$, and apply the functor $\mathrm{Hom}_{D^b(A)}(\mathrm{Ker}d^l[-l], -)$ to this distinguished triangle. Then the assertion follows from $\mathrm{Ker}d^l \in {}^\perp T$ and induction.)

Write $E = \mathrm{Ker}d^{l_0}$, and take $l = l_0$ in (2.2). By applying $\mathrm{Hom}_{D^b(A)}(E[-l_0], -)$ to (2.1) we get $h: E[-l_0] \longrightarrow Z^\bullet$ such that $s' = s \circ h$. So we have $a/s = (a \circ h)/s'$.

Apply $\mathrm{Hom}_{D^b(A)}(-, N)$ to (2.2), we get an exact sequence

$$\mathrm{Hom}_{D^b(A)}(M,N) \longrightarrow \mathrm{Hom}_{D^b(A)}(E[-l_0], N) \longrightarrow \mathrm{Hom}_{D^b(A)}(\sigma^{<l_0}T^\bullet[-1], N).$$

We claim that $\mathrm{Hom}_{D^b(A)}(\sigma^{<l_0}T^\bullet[-1], N) = \mathrm{Hom}_{D^b(A)}(\sigma^{<l_0}T^\bullet, N[1]) = 0$.

(In fact, apply $\mathrm{Hom}_{D^b(A)}(-, N[1])$ to the following distinguished triangle in $D^b(A)$

$$\sigma^{<l_0-1}T^\bullet[-1] \longrightarrow T^{l_0-1}[1-l_0] \longrightarrow \sigma^{<l_0}T^\bullet \longrightarrow \sigma^{<l_0-1}T^\bullet.$$

Then the assertion follows from induction and the assumption $N \in T^\perp$.)

Thus, there exists $f: M \longrightarrow N$ such that $f \circ s' = a \circ h$. So we have $a/s = (a \circ h)/s' = (f \circ s')/s' = f/\mathrm{Id}_M$. This shows that G is surjective.

On the other hand, if $f: M \longrightarrow N$ with $G(f) = f/\mathrm{Id}_M = 0$ in $D^b(A)/K^b(\mathrm{add}T)$, then there exists $s: Z^\bullet \Longrightarrow M$ with $\mathrm{Con}(s) \in K^b(\mathrm{add}T)$ such that $f \circ s = 0$. Use the same notation as in (2.1) and (2.2). By the argument above we have $s' = s \circ h$, and hence $f \circ s' = 0$. Therefore, by applying $\mathrm{Hom}_{D^b(A)}(-, N)$ to (2.2) we see that there exists $f': \sigma^{<l_0}T^\bullet \longrightarrow N$ such that $f' \circ \varepsilon = f$.

Consider the following distinguished triangle in $D^b(A)$

$$T^0[-1] \longrightarrow \sigma^{>0}(\sigma^{<l_0})T^\bullet \longrightarrow \sigma^{<l_0}T^\bullet \xrightarrow{\pi} T^0,$$

where $\sigma^{>0}(\sigma^{<l_0})T^\bullet = 0 \longrightarrow T^1 \longrightarrow T^2 \longrightarrow \cdots \longrightarrow T^{l_0-1} \longrightarrow 0$, and π is the natural morphism. Again since $N \in T^\perp$, it follows that $\mathrm{Hom}_{D^b(A)}(\sigma^{>0}(\sigma^{<l_0})T^\bullet, N) = 0$. By applying $\mathrm{Hom}_{D^b(A)}(-, N)$ to the above triangle we obtain an exact sequence

$$\mathrm{Hom}_{D^b(A)}(T^0, N) \longrightarrow \mathrm{Hom}_{D^b(A)}(\sigma^{<l_0}T^\bullet, N) \longrightarrow 0.$$

It follows that there exists $g : T^0 \longrightarrow N$ such that $g \circ \pi = f'$. Hence $f = g \circ (\pi \circ \varepsilon)$. Since A-mod is a full subcategory of $D^b(A)$, it follows that f factors through T^0 in A-mod. This proves that the kernel of G is $T(M, N)$, which completes the proof. ■

Consider the natural functor $\mathcal{X}_T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$, which is the composition of the embedding $\mathcal{X}_T \cap T^\perp \hookrightarrow A$-mod and the embedding A-mod $\hookrightarrow D^b(A)$, and the localization functor $D^b(A) \longrightarrow D^b(A)/K^b(\mathrm{add}T)$. Let $\underline{{}^\perp T \cap T^\perp}$ denote the stable category of ${}^\perp T \cap T^\perp$ modulo $\mathrm{add}T$.

Lemma 2.2. *Let T be a generalized cotilting module. Then the natural functor ${}^\perp T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ induces a fully faithful functor*

$$\underline{{}^\perp T \ \cap \ T^\perp} \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A).$$

Proof. Since T is generalized cotilting then $\mathcal{X}_T = {}^\perp T$, it follows that $K^b(\mathrm{add}T) = K^b(A\text{-inj})$ in $D^b(A)$. So the assertion follows from Lemma 2.1. ■

2.3. Let T be a self-orthogonal module. By Lemma 2.1 the natural functor $\underline{\mathcal{X}_T \cap T^\perp} \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ is fully faithful. It is of interest to know when it is dense.

Lemma 2.3. (i) *Assume that* inj.dim ${}_AA < \infty$. *Let T be a generalized cotilting A-module. Then the natural functor ${}^\perp T \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A)$ is dense.*

(ii) *If A is Gorenstein and T is a generalized tilting module, then the natural functor*

$$^\perp T \cap T^\perp \longrightarrow D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A) = \mathcal{D}_P(A)$$

is dense.

Proof. (i) By Happel (the dual of Lemma 4.3 in [Hap2]), the natural functor A-mod $\longrightarrow \mathcal{D}_I(A)$ is dense. So for any object $X^\bullet \in \mathcal{D}_I(A)$, there exists a module M such that $X^\bullet \cong M$ in $\mathcal{D}_I(A)$. Consider the subcategory

$$\widehat{\mathcal{X}_T} := \{M \mid \exists \text{ an exact sequence } 0 \longrightarrow X^{-n} \longrightarrow \cdots X^0 \longrightarrow M \longrightarrow 0, \ X^{-i} \in \mathcal{X}_T, \forall\, i\}.$$

Since T is generalized cotilting, it follows that $\widehat{\mathcal{X}_T} = A$-mod, and hence by Auslander and Buchweitz (Theorem 1.1 in [AB]), there exists an exact sequence

$$0 \longrightarrow Y_M \longrightarrow X_M \longrightarrow M \longrightarrow 0 \tag{2.3}$$

with $Y_M \in \mathrm{add}^\wedge T$, $X_M \in \mathcal{X}_T = {}^\perp T$. This induces a triangle $Y_M \longrightarrow X_M \longrightarrow M \longrightarrow Y_M[1]$ in $D^b(A)$, and hence a triangle in $\mathcal{D}_I(A)$. Since T is of finite injective dimension, it follows that modules in $\mathrm{add}^\wedge T$ are of finite injective dimensions, and hence $Y_M = 0$ in $\mathcal{D}_I(A)$, which implies $M \cong X_M$ in $\mathcal{D}_I(A)$. This proves (i).

(ii) Since A is Gorenstein it follows from Lemma 1.5 in [Hap2] that $D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_I(A) = \mathcal{D}_P(A)$. For any object $X^\bullet \in \mathcal{D}_P(A)$, by the dual of ($i$), there exists a module $M \in T^\perp$ such that $X^\bullet \cong M$ in $\mathcal{D}_P(A)$. Since A is Gorenstein, it follows from Lemma 1.3 in [HU] that T is also generalized cotilting. Repeat the argument in the proof of (i) and note that $\mathrm{add}^\wedge T \subseteq T^\perp$. It follows from (2.3) that $X_M \in T^\perp$, and hence $X^\bullet \cong M \cong X_M \in {}^\perp T \cap T^\perp$. This completes the proof. ■

2.4. Let T be a self-orthogonal A-module. Recall from 2.1 the category $\alpha(T)$ of T-Cohen-Macaulay modules, which is a Frobenius category and add$T$ is exactly the full subcategory of all the (relatively) projective and injective objects. Then the stable category of $\mathfrak{a}(T)$ modulo add$T$, denoted by $\underline{\mathfrak{a}(T)}$, is a triangulated category. Since $\mathfrak{a}(T)$ is a full subcategory of $\mathcal{X}_T \cap T^{\perp}$, it follows from Lemma 2.1 that we have a natural fully faithful functor $\underline{\mathfrak{a}(T)} \longrightarrow D^b(A)/K^b(\mathrm{add}T)$.

Lemma 2.4. *Let T be a self-orthogonal module. Then the natural embedding $\underline{\mathfrak{a}(T)} \longrightarrow D^b(A)/K^b(\mathrm{add}T)$ is an exact functor.*

Proof. This follows from 1.2 in [K2]. We include a direct justification.

Let $0 \longrightarrow X \xrightarrow{u} Y \xrightarrow{v} Z \longrightarrow 0$ be an exact sequence in $\mathfrak{a}(T)$, and $0 \longrightarrow X \xrightarrow{i_X} T(X) \xrightarrow{\pi_X} S(X) \longrightarrow 0$ an exact sequence with $T(X) \in \mathrm{add}T$ and $S(X) \in \mathfrak{a}(T)$. Then $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-w} S(X)$ is a distinguished triangle in $\underline{\mathfrak{a}(T)}$, where w is an A-map such that the following diagram is commutative

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, "\rho"] & Z \arrow[r] \arrow[d, "w"] & 0 \\ 0 \arrow[r] & X \arrow[r, "i_X"] & T(X) \arrow[r, "\pi_X"] & S(X) \arrow[r] & 0; \end{tikzcd} \tag{2.4}$$

Note that any distinguished triangle in $\underline{\mathfrak{a}(T)}$ is given in this way (cf. Lemma 1.2).

On the other hand, we have a distinguished triangle in $D^b(A)$

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w'} X[1] \tag{2.5}$$

with

$$w' = p_X/v' \in \mathrm{Hom}_{D^b(A)}(Z, X[1]) \tag{2.6}$$

as right fractions, where $p_X : \mathrm{Con}(u) \longrightarrow X[1]$ is the natural morphism of complexes, and $v' : \mathrm{Con}(u) \longrightarrow Z$ is the quasi-isomorphism induced by v. Denote by $p'_X : \mathrm{Con}(i_X) \longrightarrow X[1]$ the natural morphism of complexes, and $\pi'_X : \mathrm{Con}(i_X) \longrightarrow S(X)$ the quasi-isomorphism induced by π_X. Then right fraction $\beta_X := -p'_X/\pi'_X$ is in $\mathrm{Hom}_{D^b(A)}(S(X), X[1])$. We claim that $w' = -\beta_X w$ in $D^b(A)$, and hence by (2.5), $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{-\beta_X w} X[1]$ is a distinguished triangle in $D^b(A)$, and hence a distinguished triangle in $D^b(A)/K^b(\mathrm{add}T)$.

In fact, by (2.6) the claim is equivalent to $p_X = -\beta_X(wv')$ in $D^b(A)$. Denote by ρ' the chain map $\mathrm{Con}(u) \longrightarrow \mathrm{Con}(i_X)$ induced by ρ. Then $-\beta_X(wv') = (p'_X/\pi'_X)(wv') = p'_X\rho' = p_X$, where the second equality follows from the multiplication rule of right fractions and $wv = \pi_X\rho$ in (2.4).

By the distinguished triangle $X \xrightarrow{i_X} T(X) \longrightarrow \mathrm{Con}(i_X) \xrightarrow{p'_X} X[1]$ in $D^b(A)$ we get the corresponding one in $D^b(A)/K^b(\mathrm{add}T)$ with $T(X) = 0$, it follows that p'_X, and hence β_X, is an isomorphism in $D^b(A)/K^b(\mathrm{add}T)$. This shows that $\beta : G \circ S \longrightarrow [1] \circ G$ is a natural isomorphism, where G denote the natural functor $\underline{\mathfrak{a}(T)} \longrightarrow D^b(A)/K^b(\mathrm{add}T)$. This completes the proof. ■

2.5. The following result gives a relative version of the explicit description due to Happel, of the singularity categories of Gorenstein algebras (Theorem 4.6 in [Hap2]).

Theorem 2.5. *Let A be a Gorenstein algebra and T a generalized tilting A-module. Then the natural functor induces a triangle-equivalence $\underline{{}^{\perp}T \cap T^{\perp}} \cong D^b(A)/K^b(\mathrm{add}T) = \mathcal{D}_P(A) = \mathcal{D}_I(A)$.*

Proof. This follows from Lemmas 2.2, 2.3(ii) and 2.4. Note that for a Gorenstein algebra, generalized tilting modules coincide with generalized cotilting modules (Lemma 1.3 in [HU]); and in this case ${}^{\perp}T\ \cap T^{\perp} = \mathcal{X}_T \cap {}_T\mathcal{X}$. ■

Let us remark that one can also use the derived equivalence given by T to reduce Theorem 2.5 to the classical one where $T = A$ as in Theorem 4.6 of [Hap2].

2.6. The following result is different from Lemma 2.3, and seems to be of interest.

Proposition 2.6. *Assume that* inj.dim ${}_AA < \infty$. *Let* T *be a generalized tilting module. Then the natural functor* ${}^{\perp}T \cap T^{\perp} \longrightarrow \mathcal{D}_I(A)$ *is dense.*

Proof. Set $t :=$ proj.dim T. Since T is generalized tilting and inj.dim ${}_AA < \infty$, it follows that $K^b(\mathrm{add}T) = K^b(A\text{-proj}) \subseteq K^b(A\text{-inj})$, and hence inj.dim $T = s < \infty$.

Identify $D^b(A)$ with $K^{+,b}(A\text{-inj})$. For any object $I^\bullet$ in $\mathcal{D}_I(A)$, without loss of generality, we may assume that

$$I^\bullet = 0 \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{l-1} \longrightarrow I^l \xrightarrow{d^l} \cdots \longrightarrow I^{l+r-1} \longrightarrow I^{l+r} \xrightarrow{d^{l+r}} \cdots$$

with $H^n(I^\bullet) = 0$ for $n \geq l$. Set $E := \mathrm{Ker}d^{l+r}$ and $X := \mathrm{Ker}d^l$. Then the complex $0 \longrightarrow I^0 \longrightarrow \cdots \longrightarrow I^{l-1} \longrightarrow I^l \xrightarrow{d^l} \cdots \longrightarrow I^{l+r-1} \longrightarrow E \longrightarrow 0$ is quasi-isomorphic to $I^\bullet$, and hence $I^\bullet \simeq E[-(l+r)]$ in $\mathcal{D}_I(A)$.

Take $r \geq s, t$. By the exact sequence of A-modules

$$0 \longrightarrow X \longrightarrow I^l \longrightarrow \cdots \longrightarrow I^{l+r-1} \longrightarrow E \longrightarrow 0,$$

and proj.dim $T = t < \infty$ and $r \geq t$, we infer that $E \in T^{\perp}$.

By the generalized tilting theory we have $T^{\perp} = \mathrm{add}^{\sim}T = {}_T\mathcal{X}$, and hence we have an exact sequence of A-module

$$0 \longrightarrow W \longrightarrow T^{-(l+r-1)} \longrightarrow \cdots \longrightarrow T^0 \longrightarrow E \longrightarrow 0 \tag{2.7}$$

with each $T^i \in \mathrm{add}T$ and $W \in T^{\perp}$. Since $K^b(\mathrm{add}T) \subseteq K^b(A\text{-inj})$, it follows that the complex $0 \longrightarrow T^{-(l+r-1)} \longrightarrow \cdots \longrightarrow T^0 \longrightarrow 0$ is in $K^b(A\text{-inj})$, and hence $E = W[l+r]$ in $\mathcal{D}_I(A)$. Since T is self-orthogonal with inj.dim $T = s$ and $r \geq s$, by (2.7) we infer that $W \in {}^{\perp}T$. Thus $I^\bullet = W$ in $\mathcal{D}_I(A)$ with $W \in {}^{\perp}T \cap T^{\perp}$. ■

Corollary 2.7. *The following are equivalent*

(i) gl.dimA $< \infty$;

(ii) inj.dim ${}_AA < \infty$, *and* ${}^{\perp}T \cap\ T^{\perp} = \mathrm{add}T$ *for any generalized tilting module.*

Proof. The implication of $(i) \Longrightarrow (ii)$ follows from Theorem 2.5; and $(ii) \Longrightarrow (i)$ follows from Proposition 2.6, since $K^b(\mathrm{add}T) = K^b(A\text{-proj}) \subseteq K^b(A\text{-inj})$. ■

3. **Bounded derived categories of Gorenstein algebras**

3.1. Keep the notation in the introduction throughout this section, in particular for $\mathcal{N}$ and $\mathcal{M}_P$. An object in $T(A)^{\mathbb{Z}}$-mod and in $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$ is denoted by $M = \oplus_{n\in\mathbb{Z}} M_n$ with each M_n an A-module and $D(A).M_n \subseteq M_{n+1}$. The following result explicitly describes the bounded derived category of a Gorenstein algebra A inside $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$.

Theorem 3.1. *Let A be a Gorenstein algebra. Then via Happel's functor $F : D^b(A) \longrightarrow T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$ we have*

$$D^b(A) \simeq \mathcal{N} = \{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim}\ {}_A M_n < \infty,\ \forall\ n \neq 0\}$$

and

$$K^b(A\text{-proj}) \simeq \mathcal{M}_P = \{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim}\ {}_A M_n < \infty,\ \forall\ n \in \mathbb{Z}\}.$$

Corollary 3.2. *Let A be a self-injective algebra. Then*

$$D^b(A) \simeq \mathcal{N} = \{M = \oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid {}_A M_n \textit{ is projective},\ \forall\ n \neq 0\}$$

and

$$K^b(A\text{-proj}) \simeq \mathcal{M}_P = \{M = \oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid {}_A M_n \textit{ is projective},\ \forall\ n \in \mathbb{Z}\}.$$

3.2. For each $n \in \mathbb{Z}$ and an indecomposable projective A-module P, the $\mathbb{Z}$-graded $T(A)$-module

$$\text{proj}(P, n, n+1) = \oplus_{i\in\mathbb{Z}} M_i \ \text{ with } \ M_i = \begin{cases} P, & i = n; \\ D(A) \otimes_A P, & i = n+1; \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

is an indecomposable projective $\mathbb{Z}$-graded $T(A)$-module, and any indecomposable projective $\mathbb{Z}$-graded $T(A)$-module is of this form; and for an indecomposable injective A-module I, the $\mathbb{Z}$-graded $T(A)$-module

$$\text{inj}(I, n-1, n) = \oplus_{i\in\mathbb{Z}} M_i \ \text{ with } \ M_i = \begin{cases} \text{Hom}_A(D(A), I), & i = n-1; \\ I, & i = n; \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

is an indecomposable injective $\mathbb{Z}$-graded $T(A)$-module, and any indecomposable injective $\mathbb{Z}$-graded $T(A)$-module is of this form. Note that

$$\text{proj}(P, n, n+1) \simeq \text{inj}(D(A) \otimes_A P, n, n+1)$$

and

$$\text{inj}(I, n-1, n) \simeq \text{proj}(\text{Hom}_A(D(A), I), n-1, n).$$

Any homogeneous $\mathbb{Z}$-graded $T(A)$-module $M = M_n$ of degree n has the injective hull $\text{inj}(I_A(M_n), n-1, n)$, and the projective cover $\text{proj}(P_A(M_n), n, n+1)$, where $I_A(M_n)$ and $P_A(M_n)$ are respectively the injective hull and the projective cover of M_n as an A-module. See [Hap1], II. 4.1.

Lemma 3.3. *Let A be a Gorenstein algebra. Then the full subcategories given by*

$$\{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim}\ {}_A M_n < \infty,\ \forall\ n \neq 0\}$$

and

$$\{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim}\ {}_A M_n < \infty,\ \forall\ n \in \mathbb{Z}\}$$

are triangulated subcategories of $T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}}$.

Proof. Since A is Gorenstein, it follows from (3.2) that the two subcategories above contain all the injective modules in $T(A)^{\mathbb{Z}}$-mod. Given a short exact sequence $0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$ in $T(A)^{\mathbb{Z}}$-mod, then for each n we have an exact sequence of A-modules $0 \longrightarrow M_n \longrightarrow N_n \longrightarrow L_n \longrightarrow 0$. Note that if any two terms of the short exact sequence above have finite projective dimensions, then the other one also has finite projective dimension. Now the assertion follows from Lemma 1.3. ■

3.3. **Proof of Theorem 3.1.** We only prove

$$\mathcal{N} = \{\oplus_{n\in\mathbb{Z}} M_n \in T(A)^{\mathbb{Z}}\text{-}\underline{\mathrm{mod}} \mid \text{proj.dim}\ {}_A M_n < \infty,\ \forall\ n \neq 0\}. \tag{3.3}$$

The other equality can be similarly proven. Since A is Gorenstein, it follows that

$$\{\oplus_{n\in\mathbb{Z}} M_n \mid \text{proj.dim}\ {}_A M_n < \infty,\ \forall\ n \neq 0\} = \{\oplus_{n\in\mathbb{Z}} M_n \mid \text{inj.dim}\ {}_A M_n < \infty,\ \forall\ n \neq 0\}.$$

By Lemma 3.3 the right hand side in (3.3) is a triangulated subcategory of $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$ containing all the A-modules, while by definition $\mathcal{N}$ is the triangulated subcategory of $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$ generated by A-mod. It follows that $\mathcal{N} \subseteq \{M = \oplus_{n\in\mathbb{Z}} M_n \mid \text{proj.dim}\ {}_A M_n < \infty,\ \forall\ n \neq 0\}$.

For the other inclusion, first, consider all the objects of the form $M = \oplus_{i\geq 0} M_i$ in the right hand side of (3.3). We claim that such an M lies in $\mathcal{N}$, by using induction on $l(M) := max\{\ i \mid M_i \neq 0\}$. Assume that $\mathcal{N}$ already contains all such objects M with $l(M) < n,\ \ n \geq 1$. Now, we use induction on $m :=$ inj.dim ${}_A M_n$ to prove that $M = \oplus_{i=0}^{n} M_i \in \mathcal{N}$, where inj.dim ${}_A M_i < \infty,\ \forall\ i \neq 0$.

If $m = 0$, i.e., M_n is injective as an A-module, then by the exact sequences in $TA^{\mathbb{Z}}$-mod

$$0 \longrightarrow M_n \longrightarrow M \longrightarrow M/M_n \longrightarrow 0. \tag{3.4}$$

and (see (3.2))

$$0 \longrightarrow M_n \longrightarrow \text{inj}(M_n, n-1, n) \longrightarrow M_n[1] \longrightarrow 0,$$

we have by induction $M/M_n,\ M_n[1] \in \mathcal{N}$, and hence $M_n \in \mathcal{N}$. By (3.4) we have the distinguished triangle in $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$

$$M_n \longrightarrow M \longrightarrow M/M_n \longrightarrow M_n[1] \tag{3.5}$$

with $M_n,\ M/M_n \in \mathcal{N}$. It follows from $\mathcal{N}$ being a triangulated subcategory that $M \in \mathcal{N}$.

Assume that for $n, d \geq 1$, $\mathcal{N}$ already contains all the objects $M = \oplus_{i=0}^{n} M_i$ in the right hand side of (3.3) with inj.dim ${}_A M_n < d$. We will prove that $\mathcal{N}$ also contains such an object M with inj.dim ${}_A M_n = d$. Take an exact sequence in $T(A)^{\mathbb{Z}}$-mod (see (3.2))

$$0 \longrightarrow M_n \longrightarrow \text{inj}(I_A(M_n), n-1, n) \longrightarrow M_n[1] \longrightarrow 0.$$

Since the n-th component $I_A(M_n)/M_n$ of $M_n[1]$ has injective dimension less than d, it follows from induction that $M_n[1] \in \mathcal{N}$, and hence $M_n \in \mathcal{N}$. Also $M/M_n \in \mathcal{N}$ since $l(M/M_n) < n$. Thus $M \in \mathcal{N}$ by (3.5). This proves the claim.

Dually, any object of the form $M = \oplus_{i\leq 0} M_i$ in the right hand side of (3.3) lies in $\mathcal{N}$.

In general, for $M = \oplus_{n\in\mathbb{Z}} M_n$ in the right hand side of (3.3), set $M_{\geq 0} := \oplus_{n\geq 0} M_n$. Then it is a submodule of M. By the argument above we have $M_{\geq 0} \in \mathcal{N}$ and $M/M_{\geq 0} \in \mathcal{N}$. Since the exact sequence in $T(A)^{\mathbb{Z}}$-mod $0 \longrightarrow M_{\geq 0} \longrightarrow M \longrightarrow M/M_{\geq 0} \longrightarrow 0$ induces a distinguished triangle in $T(A)^{\mathbb{Z}}$-$\underline{\mathrm{mod}}$, and $\mathcal{N}$ is a triangulated subcategory, it follows that $M \in \mathcal{N}$. This completes the proof. ∎

Appendix: Stable category associated with self-orthogonal modules

We include a description of the stable category $\underline{\mathfrak{a}(T)}$ of the Frobenius category $\mathfrak{a}(T) := \mathcal{X}_T \cap {}_T\mathcal{X}$ of T-Cohen-Macaulay modules (cf. 2.1), where T is a self-orthogonal A-module. Denote by $K^{ac}(T)$ the full subcategory of the unbounded homotopy category $K(A)$ consisting of acyclic complexes with components in add$T$. It is a triangulated subcategory. Then we have

Theorem A.1. *Let T be a self-orthogonal module such that* $\mathrm{add}^{\sim}T \subseteq T^{\perp}$ *and* $\mathrm{add}_{\sim}T \subseteq {}^{\perp}T$. *Then there is a triangle-equivalence* $\underline{\mathfrak{a}(T)} \simeq K^{ac}(T)$.

We provide a direct proof. It can be also deduced by using Theorem 3.11 in [Bel]. Together with Theorem 2.5, we get another description of the singularity category of a Gorenstein algebra. For a similar result on separated noetherian schemes see [Kr], Theorem 1.1(3).

Corollary A.2. *Let A be a Gorenstein algebra and T be a generalized tilting module. Then we have triangle-equivalences* $\mathcal{D}_I(A) \overset{\sim}{\longleftarrow} K^{ac}(T) \overset{\sim}{\longrightarrow} \mathcal{D}_P(A)$.

A.1. Let $X \in {}_T\mathcal{X}$ with exact sequence

$$\cdots \longrightarrow T^{-i} \xrightarrow{d_T^{-i}} T^{-(i-1)} \longrightarrow \cdots \xrightarrow{d_T^{-1}} T^0 \xrightarrow{d_T^0} X \longrightarrow 0,$$

where each $T^{-i} \in \mathrm{add}T$ and $\mathrm{Ker}d^{-i} \in T^{\perp}$, $i \geq 0$. Let $Y \in A$-mod with a complex

$$\cdots \longrightarrow T'^{-i} \xrightarrow{d_{T'}^{-i}} T'^{-(i-1)} \longrightarrow \cdots \xrightarrow{d_{T'}^{-1}} T'^0 \xrightarrow{d_{T'}^0} Y \longrightarrow 0,$$

where each $T'^{-i} \in \mathrm{add}T$. Denote them by $T^\bullet \xrightarrow{d_T^0} X$ and $T'^\bullet \xrightarrow{d_{T'}^0} Y$, respectively.

The proof of the following fact is similar with the one of the Comparison-Theorem in homological algebra.

Lemma A.3. *With the notation of $X, Y, T^\bullet, T'^\bullet$ as above, for any morphism $f : Y \longrightarrow X$, there exists a unique morphism $f^\bullet : T'^\bullet \longrightarrow T^\bullet$ in $K(A)$ such that $fd_{T'}^0 = d_T^0 f^0$.*

The following fact is in p.446 in [Ric1], or p.45 in [KZ].

Lemma A.4. *Let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be a full and exact functor of triangulated categories. Then F is faithful if and only if it is faithful on objects, that is, if $F(X) \simeq 0$ then $X \simeq 0$.*

A.2. **Proof of Theorem A.1.** Since $\mathrm{add}^{\sim}T \subseteq T^{\perp}$ and $\mathrm{add}_{\sim}T \subseteq {}^{\perp}T$, it follows that ${}_T\mathcal{X} = \mathrm{add}^{\sim}T$ and $\mathcal{X}_T = \mathrm{add}_{\sim}T$. Thus, for any object $T^\bullet$ in $K^{ac}(T)$ we have $\mathrm{Coker}d_T^i \in \mathrm{add}^{\sim}T \cap \mathrm{add}_{\sim}T = \mathfrak{a}(T)$, for each $i \in \mathbb{Z}$.

Define a functor $F : K^{ac}(T) \longrightarrow \underline{\mathfrak{a}(T)}$ as follows: for an object $T^\bullet$ in $K^{ac}(T)$, define $F(T^\bullet) := \mathrm{Coker}d_T^{-1}$; for a morphism $f^\bullet : T^\bullet \longrightarrow T'^\bullet$ in $K^{ac}(T)$, define $F(f^\bullet)$ to be the image in $\underline{\mathfrak{a}(T)}$ of the unique morphism $\bar{f}^0 : \mathrm{Coker}d_T^{-1} \longrightarrow \mathrm{Coker}d_{T'}^{-1}$ induced by f^0. Note that F is well-defined, dense, and full by Lemma A.3 and its dual.

Note that F is faithful on objects. In fact, if $F(T^\bullet) \simeq 0$, then $\mathrm{Coker}d_T^{-1} \in \mathrm{add}T$. Since $\mathrm{Coker}d_T^i \in \mathrm{add}^{\sim}T \cap \mathrm{add}_{\sim}T \subseteq T^{\perp} \cap {}^{\perp}T$ for each i, it follows that the exact sequence $0 \longrightarrow \mathrm{Coker}d_T^{-2} \longrightarrow T^0 \longrightarrow \mathrm{Coker}d_T^{-1} \longrightarrow 0$ splits, and hence $\mathrm{Coker}d_T^{-2} \in \mathrm{add}T$. Repeating this process, we have the split exact sequence $0 \longrightarrow \mathrm{Coker}d_T^{-i-2} \longrightarrow T^{-i} \longrightarrow \mathrm{Coker}d_T^{-i-1} \longrightarrow 0$, and $\mathrm{Coker}d_T^{-i-2} \in \mathrm{add}T$, for each $i \geq 0$. Similarly, the exact sequence $0 \longrightarrow \mathrm{Coker}d_T^{i-2} \longrightarrow T^i \longrightarrow \mathrm{Coker}d_T^{i-1} \longrightarrow 0$ splits and $\mathrm{Coker}d_T^{i-1} \in \mathrm{add}T$ for each $i \geq 1$. This implies that the identity $\mathrm{Id}_{T^\bullet}$ is homotopic to zero, that is, $T^\bullet$ is zero in $K^{ac}(T)$.

In order to prove that F is an exact functor, we first need to establish a natural isomorphism $F \circ [1] \longrightarrow [1] \circ F$, where the first $[1]$ is the usual shift of complexes, and the second $[1]$ is the shift functor of the stable category $\underline{\mathfrak{a}(T)}$. In fact, for each $T^\bullet \in K^{ac}(T)$, we have a commutative diagram of short exact sequences in A-mod

$$\begin{tikzcd}
0 \arrow[r] & F(T^\bullet) \arrow[r, "i_{T^\bullet}"] \arrow[d, equal] & T^1 \arrow[r, "\pi_{T^\bullet}"] \arrow[d, "\gamma_{T^\bullet}"] & F(T^\bullet[1]) \arrow[r] \arrow[d, "\alpha_{T^\bullet}"'] & 0 \\
0 \arrow[r] & F(T^\bullet) \arrow[r, "i_{F(T^\bullet)}"] & T(F(T^\bullet)) \arrow[r, "\pi_{F(T^\bullet)}"] & F(T^\bullet)[1] \arrow[r] & 0
\end{tikzcd}$$

where $i_{T^\bullet}$ is the natural embedding, $\pi_{T^\bullet}$ is the canonical map, and the second row is the one defining $F(T^\bullet)[1]$, with $T(F(T^\bullet)) \in \mathrm{add}T$ (see (1.1)). Note that $\underline{\alpha_{T^\bullet}}$ is unique in the stable category $\underline{\mathfrak{a}(T)}$, and that it is easy to verify that $\underline{\alpha} : F \circ [1] \longrightarrow [1] \circ F$ is a natural isomorphism (by using the same argument as in the proof of Lemma 2.2 in [Hap1], p.12). We will show that $(F, \underline{\alpha})$ is an exact functor.

Consider a distinguished triangle in $K^{ac}(T)$ by mapping cone

$$T^\bullet \xrightarrow{f^\bullet} T'^\bullet \xrightarrow{\binom{0}{1}} \mathrm{Con}(f^\bullet) \xrightarrow{(1,\ 0)} T^\bullet[1].$$

Write $\theta := F(\binom{0}{1})$ and $\eta := F((1,0))$. Clearly we have $\eta\theta = 0$. Observe that the sequence in A-mod

$$0 \longrightarrow F(T^\bullet) \xrightarrow{\binom{\bar{f}^0}{-i_{T^\bullet}}} F(T'^\bullet) \oplus T^1 \xrightarrow{(\theta,\ \pi)} F(\mathrm{Con}(f^\bullet)) \longrightarrow 0,$$

is exact, where π is the natural map from T^1 to $F(\mathrm{Con}(f^\bullet)) = (T^1 \oplus T'^0)/\mathrm{Im}\begin{pmatrix} -d_T^0 & 0 \\ -f^0 & d_{T'}^{-1} \end{pmatrix}$. This can be seen by directly verifying that (θ, π) is surjective, $(\theta, \pi)\binom{\bar{f}^0}{-i_{T^\bullet}} = 0$, and $\mathrm{Ker}(\theta, \pi) \subseteq \mathrm{Im}\binom{\bar{f}^0}{-i_{T^\bullet}}$. By definition we have $\eta\pi = \pi_{T^\bullet}$, and hence the following diagram of short exact sequences in A-mod commutes

$$\begin{tikzcd}
0 \arrow[r] & F(T^\bullet) \arrow[r, "\binom{\bar{f}^0}{-i_{T^\bullet}}"] \arrow[d, equal] & F(T'^\bullet) \oplus T^1 \arrow[r, "(\theta,\ \pi)"] \arrow[d, "(0, -\gamma_{T^\bullet})"] & F(\mathrm{Con}(f^\bullet)) \arrow[r] \arrow[d, "-(\alpha_{T^\bullet}\eta)"] & 0 \\
0 \arrow[r] & F(T^\bullet) \arrow[r, "i_{F(T^\bullet)}"] & T(F(T^\bullet)) \arrow[r, "\pi_{F(T^\bullet)}"] & F(T^\bullet)[1] \arrow[r] & 0.
\end{tikzcd}$$

It follows from Lemma 1.2 that $F(T^\bullet) \xrightarrow{F(f^\bullet)} F(T'^\bullet) \xrightarrow{\theta} F(\mathrm{Con}(f^\bullet)) \xrightarrow{\underline{\alpha_{T^\bullet}\eta}} F(T^\bullet)[1]$ is a distinguished triangle in $\underline{\mathfrak{a}(T)}$. This proves that $F : K^{ac}(T) \longrightarrow \underline{\mathfrak{a}(T)}$ is an exact functor.

By Lemma A.4 the proof is completed. ■

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References

[AB] M. Auslander and R. O. Buchweitz, The homological theory of maximal Cohen-Macaulay approximations, Mémoires de la Soc. Math. France 38(1989), 5-37.

[AR1] M. Auslander and I. Reiten, Applications of contravariantly finite subcategories, Adv. Math. 86 (1)(1991), 111-152.

[AR2] M. Auslander and I. Reiten, Cohen-Macaulay and Gorenstein artin algebras, In: Representation theory of finite groups and finite-dimensional algebras (Proc. Conf. at Bielefeld, 1991), 221–245, Progress in Math. vol. 95, Birkhäuser, Basel, 1991.

[BBD] A. Beilinson, J. Bernstein, and P. Deligne, Faisceaux pervers, Astérisque 100, Societé Mathématique de France, Paris, 1982.

[BD] I. Burban and Yu. A. Drozd, Derived categories of nodal algebras, J. Algebra 272(1)(2004), 46-94.

[Bel] A. Beligiannis, The homological theory of contravariantly finite subcategories: Auslander-Buchweitz contexts, Gorenstein categories and (co)stabilization, Commun. Algebra 28(10)(2000), 4547-4596.

[Buc] R. O. Buchweitz, Maximal Cohen-Macaulay modules and Tate cohomology over Gorenstein rings, Unpublished manuscript, Hamburg (1987), 155pp.
[CPS] E. Cline, B. Parshall, and L. Scott, Derived categories and Morita theory, J. Algebra 104(1986), 397-409.
[Hap1] D. Happel, Triangulated categories in representation theory of finite dimensional algebras, London Math. Soc. Lecture Notes Ser. 119, Cambridge Uni. Press, 1988.
[Hap2] D. Happel, On Gorenstein algebras, In: Representation theory of finite groups and finite-dimensional algebras (Proc. Conf. at Bielefeld, 1991), 389-404, Progress in Math., vol. 95, Birkhäuser, Basel, 1991.
[Hap3] D. Happel, Auslander-Reiten triangles in derived categories of finite-dimensional algebras, Proc. Amer. Math. Soc. 112(3)(1991), 641–648.
[Har] R. Hartshorne, Residue and duality, Lecture Notes in Math. 20, Springer-Verlag, 1966.
[HR] D. Happel and C. M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. 274(2)(1982), 399-443.
[HU] D. Happel and L. Unger, Modules of finite projective dimension and cocovers, Math. Ann. 306(1996), 445-457.
[I] B. Iversen, Cohomology of sheaves, Springer-Verlag, 1980.
[J] P. Jøgensen, Spectra of modules, J. Algebra 244(2)(2001), 744-784.
[K1] B. Keller, Chain complexes and stable categories, Manuscripta Math. 67 (1990), 379-417.
[K2] B. Keller, Derived categories and universal problems, Commun. Algebra 19(3)(1991), 699-747.
[KR] B. Keller and I. Reiten, Cluster-tilted algebras are Gorenstein and stably Calabi-Yau, arXiv:math.RT/0512471.
[KV1] B. Keller and D. Vossieck, Sous les catégories dérivées, C. R. Acad. Sci. Paris 305 (1987), 225-228.
[KV2] B. Keller and D. Vossieck, Aisles in derived categories, Bull. Soc. Math. Belgium 40 (1988), 239-253.
[Kr] H. Krause, The stable derived category of a noetherian scheme, Compos. Math. 141 (2005), 1128-1162.
[KI] H. Krause and S. Iyengar, Acyclicity versus total acyclicity for complexes over noetherian rings, Documenta Math. 11 (2006), 207-240.
[KZ] S. König and A. Zimmermann, Derived equivalences for group rings, Lecture Notes in Math. 1685, Springer-Verlag, 1998.
[M] T. Miyashita, Tilting modules of finite projective dimension, Math. Z. 193 (1986), 113-146.
[O1] D. Orlov, Triangulated categories of singularities and D-branes in Landau-Ginzburg models, Proc. Steklov Inst. Math. 246(3)(2004), 227–248.
[O2] D. Orlov, Derived categories of coherent sheaves and triangulated categories of singularities, arXiv:math.AG/0503632.
[Q] D. Quillen, Higher algebraic K-theory I, In: Lecture Notes in Math. 341, 85-147, Springer-Verlag, 1973.
[Ric1] J. Rickard, Morita theory for derived categories, J. London Math. Soc. 39(2)(1989), 436-456.
[Ric2] J. Rickard, Derived categories and stable equivalence, J. Pure and Applied Algebra 61 (1989), 303-317.
[Rin1] C. M. Ringel, Tame algebras and integral quadratic forms, Lecture Notes in Math. 1099, Springer-Verlag, 1984.
[Rin2] C. M. Ringel, The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences, Math. Z. 208(1991), 209-225.
[V1] J. L. Verdier, Catégories dérivées, etat 0, Springer Lecture Notes 569 (1977), 262-311.
[V2] J. L. Verdier, Des catégories dérivées des catégories abéliennes, Astérisque 239, Societé Mathématique de France, Paris, 1996.